Arithmetic in vanilla λ-calculus

Constants

$$
\begin{aligned}
\text{“zero”} &\coloneqq \lambda f\ .\ \lambda x\ .\ x \\
\text{“one”} &\coloneqq \lambda f\ .\ \lambda x\ .\ fx \\
\text{“two”} &\coloneqq \lambda f\ .\ \lambda x\ .\ f(fx) \\
\text{“three”} &\coloneqq \lambda f\ .\ \lambda x\ .\ f(f(fx)) \\
\text{“four”} &\coloneqq \lambda f\ .\ \lambda x\ .\ f(f(f(fx))) \\
&\ldots
\end{aligned}
$$

Successor

Definition.

$$\text{SUCC} \coloneqq \lambda n\ .\ \lambda f\ .\ \lambda x\ .\ f(nfx)$$

Unpacked.

$$\text{SUCC}\ n = \lambda f\ .\ \lambda x\ .\ f(nfx)$$

Example $(3 + 1 = 4)$.

$$
\begin{aligned}
\text{SUCC “three”} &= \lambda f\ .\ \lambda x\ .\ f(\text{“three”} f\ x) \\
&= \lambda f\ .\ \lambda x\ .\ f(\ (\lambda f\ .\ \lambda x\ .\ f(f(fx)))\ f\ x) \\
&= \lambda f\ .\ \lambda x\ .\ f(\ (\lambda x\ .\ f(f(fx)))\ x) \\
&= \lambda f\ .\ \lambda x\ .\ f(\ f(f(fx))\ ) \\
&= \text{“four”}
\end{aligned}
$$

Addition

Definition.

$$\text{PLUS} \coloneqq \lambda m\ .\ \lambda n\ .\ \lambda f\ .\ \lambda x\ .\ mf(nfx)$$

Unpacked.

$$\text{PLUS}\ m\ n = \lambda f\ .\ \lambda x\ .\ mf(nfx)$$

Example $(3+2=5)$.

$$\begin{aligned}
\text{PLUS “three” “two”} &= \lambda f\ .\ \lambda x\ .\ \text{“three”} f(\text{“two”} fx) \\
&= \lambda f\ .\ \lambda x\ .\ (\lambda f\ .\ \lambda x\ .\ f(f(fx)))\ f(\text{“two”} fx) \\
&= \lambda f\ .\ \lambda x\ .\ (\lambda x\ .\ f(f(fx)))\ (\text{“two”} fx) \\
&= \lambda f\ .\ \lambda x\ .\ f(f(f\ \ (\text{“two”} fx)\ )) \\
&= \lambda f\ .\ \lambda x\ .\ f(f(f\ \ (\ (\lambda f\ .\ \lambda x\ .\ f(fx))\ fx)\ )) \\
&= \lambda f\ .\ \lambda x\ .\ f(f(f\ \ (\ (\lambda x\ .\ f(fx))\ x)\ )) \\
&= \lambda f\ .\ \lambda x\ .\ f(f(f\ \ (f(fx))\ )) \\
&= \text{“five”}
\end{aligned}$$

Multiplication

Definition.

$$\text{MULT} \coloneqq \lambda m\ .\ \lambda n\ .\ \lambda f\ .\ m(nf)$$

Unpacked.

$$\text{MULT}\ m\ n = \lambda f\ .\ m(nf)$$

Example $(3 \cdot 2 = 6)$.

$$\begin{aligned}
\text{MULT “three” “two”} &= \lambda f\ .\ \text{“three”}(\text{“two”} f) \\
&= \lambda f\ .\ (\lambda f\ .\ \lambda x\ .\ f(f(fx)))\ ((\lambda f\ .\ \lambda x\ .\ f(fx))\ f) \\
&= \lambda f\ .\ (\lambda g\ .\ \lambda y\ .\ g(g(gy)))\ ((\lambda h\ .\ \lambda z\ .\ h(hz))\ f) \\
&= \lambda f\ .\ (\lambda g\ .\ \lambda y\ .\ g(g(gy)))\ (\lambda z\ .\ f(fz)) \\
&= \lambda f\ .\ \lambda y\ .\ \ (\lambda z\ .\ f(fz))\ (\ (\lambda z\ .\ f(fz))\ (\ (\lambda z\ .\ f(fz))\ y)) \\
&= \lambda f\ .\ \lambda y\ .\ \ (\lambda a\ .\ f(fa))\ (\ (\lambda b\ .\ f(fb))\ (\ (\lambda c\ .\ f(fc))\ y)) \\
&= \lambda f\ .\ \lambda y\ .\ \ (\lambda a\ .\ f(fa))\ (\ (\lambda b\ .\ f(fb))\ (f(fy))) \\
&= \lambda f\ .\ \lambda y\ .\ \ (\lambda a\ .\ f(fa))\ (f(f\ \ (f(fy))\ )) \\
&= \lambda f\ .\ \lambda y\ .\ f(f\quad (f(f\ \ (f(fy))\ ))\quad) \\
&= \text{“six”}
\end{aligned}$$

Exponentiation

Definition.

$$\text{POWN} \coloneqq \lambda m \,.\, \lambda n \,.\, n\; m$$

Unpacked.

$$\text{POWN}\; m\; n = n\; m$$

Example ($3^2 = 9$).

$$\begin{aligned}
\text{POWN “three” “two”} &= \text{“two” “three”} \\
&= (\lambda f \,.\, \lambda x \,.\, f(fx)) \;\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \\
&= (\lambda g \,.\, \lambda y \,.\, g\,(g\,y)) \;\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \\
&= \lambda y \,.\, (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \\
&= \lambda y \,.\, (\lambda h \,.\, \lambda z \,.\, h(h(h\,z))) \; (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \\
&= \lambda y \,.\, (\lambda z \,. \quad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad (\\
&\qquad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad (\quad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad z) \\
&\quad)) \\
&= \lambda y \,.\, (\lambda z \,. \quad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad (\\
&\qquad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad (\quad (\lambda x \,.\, y(y(yx))) \quad z) \\
&\quad)) \\
&= \lambda y \,.\, (\lambda z \,. \quad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad (\\
&\qquad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad (y(y(yz))) \\
&\quad)) \\
&= \lambda y \,.\, (\lambda z \,. \quad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad (\\
&\qquad (\lambda x \,.\, y(y(yx))) \quad (y(y(yz))) \\
&\quad)) \\
&= \lambda y \,.\, (\lambda z \,. \quad (\; (\lambda f \,.\, \lambda x \,.\, f(f(fx))) \; y) \quad (\quad y(y(y \;\; (y(y(yz))) \;\;)) \quad)) \\
&= \lambda y \,.\, (\lambda z \,. \quad (\; \lambda x \,.\, y(y(yx))) \quad (\quad y(y(y \;\; (y(y(yz))) \;\;)) \quad)) \\
&= \lambda y \,.\, (\lambda z \,. \quad (y(y(y \quad (\quad y(y(y \;\; (y(y(yz))) \;\;)) \quad))) \quad) \quad) \\
&= \lambda f \,.\, (\lambda x \,. \quad (f(f(f \quad (\quad f(f(f \;\; (f(f(fx))) \;\;)) \quad))) \quad) \quad) \\
&= \lambda f \,.\, \lambda x \,. \quad f(f(f \quad (\quad f(f(f \;\; (f(f(fx))) \;\;)) \quad))) \\
&= \text{“nine”}
\end{aligned}$$

Predecessor

Definition.

$$\text{PRED} \coloneqq \lambda n\ .\ \lambda f\ .\ \lambda x\ .\ n\ (\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ (\lambda u\ .\ u)$$

Unpacked.

$$\text{PRED}\ n = \lambda f\ .\ \lambda x\ .\ n\ (\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ (\lambda u\ .\ u)$$

Example $(3 - 1 = 2)$.

$$\begin{aligned}
\text{PRED “three”} &= \lambda f\ .\ \lambda x\ .\ \text{“three”}\ (\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ (\lambda u\ .\ u) \\
&= \lambda f\ .\ \lambda x\ .\ (\lambda f\ .\ \lambda x\ .\ f(f(fx)))\ (\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ (\lambda u\ .\ u) \\
&= \lambda f\ .\ \lambda x\ .\ (\lambda x\ .\ (\lambda g\ .\ \lambda h\ .\ h(gf))((\lambda g\ .\ \lambda h\ .\ h(gf))((\lambda g\ .\ \lambda h\ .\ h(gf))x))) \\
&\quad (\lambda u\ .\ x)\ (\lambda u\ .\ u) \\
&= \lambda f\ .\ \lambda x\ .\ ((\lambda g\ .\ \lambda h\ .\ h(gf))((\lambda g\ .\ \lambda h\ .\ h(gf))((\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ ))) \\
&\quad (\lambda u\ .\ u) \\
&= \lambda f\ .\ \lambda x\ .\ (\lambda h\ .\ h(\quad ((\lambda g\ .\ \lambda h\ .\ h(gf))((\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ ))\quad f)) \\
&\quad (\lambda u\ .\ u) \\
&= \lambda f\ .\ \lambda x\ .\ (\lambda u\ .\ u)(\quad ((\lambda g\ .\ \lambda h\ .\ h(gf))((\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ ))\quad f) \\
&= \lambda f\ .\ \lambda x\ .\ ((\lambda g\ .\ \lambda h\ .\ h(gf))((\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ ))\ \ f \\
&= \lambda f\ .\ \lambda x\ .\ (\lambda h\ .\ h(\quad ((\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ )\quad f))\ \ f \\
&= \lambda f\ .\ \lambda x\ .\ f(\quad ((\lambda g\ .\ \lambda h\ .\ h(gf))\ (\lambda u\ .\ x)\ )\quad f) \\
&= \lambda f\ .\ \lambda x\ .\ f(\quad ((\lambda h\ .\ h(\ (\lambda u\ .\ x)\ f)))\quad f) \\
&= \lambda f\ .\ \lambda x\ .\ f(\quad (\lambda h\ .\ hx)\quad f) \\
&= \lambda f\ .\ \lambda x\ .\ f(\quad f\quad x\quad) \\
&= \text{“two”}
\end{aligned}$$